Lecture 15: Cohomology of Noetherian Affine Schemes

Note Title 11/2/2019

Theorem 1 (Serre) X: Noetherian scheme

TFAE ① X affine

② $H^i(X, \mathcal{F}) = 0$, $\forall i > 0$, $\mathcal{F}$: quasi-coherent

③ $H^i(X, \mathcal{F}) = 0$, $\forall i > 0$, $\mathcal{F}$: coherent ideal

Remark: $/\mathbb{C}$ is true for affine replaced by Stein.

Key lemma: I injective module over a Noetherian ring A

$\Longrightarrow \tilde{I}$ is a flasque sheaf on $X = \operatorname{Spec} A$.

Theorem 2: $X = \operatorname{Spec} A$. A: Noetherian

$\mathcal{F}$: quasi-coherent sheaf on X

$\Longrightarrow H^i(X, \mathcal{F}) = 0$, $\forall i > 0$

pf: $M = \Gamma(X, \mathcal{F})$, w/ injective resolution $0 \to M \to I^\bullet$ A-modules

$$\rightsquigarrow \quad 0 \to \tilde{M} \to \tilde{I}^\bullet \quad \text{flasque resolution}$$

$\tilde{M} = \mathcal{F}$

can calculate the cohomology

Γ

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \tilde{I}^\bullet) \quad \text{exact}$$

$\Gamma(X, \mathcal{F}) = M$, $\Gamma(X, \tilde{I}^\bullet) = I^\bullet$

$$\Longrightarrow H^i(X, \mathcal{F}) = 0, \ \forall i > 0$$

(Proof of Theorem 1) ① $\Longrightarrow$ ② $\Longrightarrow$ ③ ✓

$\uparrow$ Theorem 2

③ $\Longrightarrow$ ①

Lemma 1: A scheme X is affine iff $\exists f_1, \cdots, f_r \in A = \Gamma(X, \mathcal{O}_X)$

s.t X_{f_i} affine & $f_1, \cdots, f_r$ generate $1 \in A$

$\forall p \in X$, choose an affine open set $\underset{p \in}{U} \subseteq X$, $Y = X \setminus U$, $U \cong \operatorname{Spec} A$

$$0 \to \mathcal{J}_{Y \cup \{p\}} \to \mathcal{J}_Y \to \underset{\overset{\cong}{\mathcal{O}_p / m_p}}{k(p)} \to 0$$

$$\Rightarrow \Gamma(X, \mathcal{J}_{Y \cup \{p\}}) \longrightarrow \Gamma(X, \mathcal{J}_Y) \twoheadrightarrow \Gamma(X, k(p)) \longrightarrow \underset{\overset{\parallel}{0}}{H^1(X, \mathcal{J}_{Y \cup \{p\}})}$$

i.e $\exists f \in \Gamma(X, \mathcal{J}_Y) \subseteq \Gamma(X, \mathcal{O}_X)$ s.t $f_p \notin m_p$

$$X_f = \{ x \in X \mid f_x \notin m_x \} \subseteq U \cong \operatorname{Spec} A$$

$f \in \Gamma(X, \mathcal{J}_Y)$

$f|_Y \equiv 0$

$$\Longrightarrow \underset{\text{affine}}{X_f \cong \operatorname{Spec} A_{f'}}, \quad \begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(U, \mathcal{O}_U) \\ f & \longmapsto & f' \end{array}$$

In other words, every point there is an affine neighborhood

X_f, $f \in \Gamma(X, \mathcal{O}_X)$

X Noetherian $\Longrightarrow$ $\exists$ finite affine open cover X_{f_i}, $i = 1, \cdots, r$

thus quasi-compact

It suffices to prove that f_i generate $1 \in \Gamma(X, \mathcal{O}_X)$

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X \longrightarrow 0 \qquad \because X_{f_i} \text{ cover } X$$

$$a_i \longmapsto \Sigma a_i f_i$$

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}_X^{r} \supseteq \mathcal{F} \cap \mathcal{O}_X^{r-1} \supseteq \cdots \supseteq \mathcal{F} \cap \mathcal{O}_X$$

successive quotients are coherent sheaf $\Rightarrow H^1(X, \mathcal{F}) = 0$

$$\therefore \quad \Gamma\left(X, \bigoplus_{i=1}^{r} \mathcal{O}_X\right) \twoheadrightarrow \Gamma(X, \mathcal{O}_X) = A$$

$$\begin{array}{ccc} \cup & & \cup \\ a_i & \longmapsto & \sum_i a_i f_i = 1 \end{array}$$

Proof of Lemma:

Recall $\quad \mathrm{Hom}_{\text{ring}}(A, \Gamma(X, \mathcal{O}_X)) \cong \mathrm{Hom}_{Sch}(X, \operatorname{Spec} A)$

$$\mathrm{id} \longmapsto (X \xrightarrow{f} \operatorname{Spec} A)$$

Take $A = \Gamma(X, \mathcal{O}_X)$

Notice that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$

- $a \in \Gamma(X, \mathcal{O}_X),\ a|_{X_f} = 0 \Rightarrow \exists n,\ f^n a = 0$
- $b \in \Gamma(X_f, \mathcal{O}_{X_f}) \Rightarrow \exists n.\ f^n b \in \Gamma(X, \mathcal{O}_X)$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & \Gamma(X, \mathcal{O}_X) \arrow[d, "res"] \\ A_{f_i} \arrow[r, "\cong"] & \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \end{tikzcd} \quad \approx \quad \begin{tikzcd} X \arrow[r, "f"] & \operatorname{Spec} A \\ \underset{\text{affine}}{X_{f_i}} \arrow[u, hook] \arrow[r, "\cong"] & \operatorname{Spec} A_{f_i} \arrow[u, hook] \end{tikzcd}$$

f_i generate $1 \in A \Rightarrow X_{f_i}$ affine open cover of X

$$\Longrightarrow X \cong \operatorname{Spec} A$$